AD-

17942

UNCLASSIFIED

OFFICE OF NAVAL RESEARCH

Contract Nonr-562(05)

Technical Report No. 1

Some Statistical Properties of Gated Signal Plus Narrow Band Noise

by

L. C. Maximon and J. P. Ruina

(Brown University)

DIVISION OF ENGINEERING

BROWN UNIVERSITY

PROVIDENCE, R. I.

August 1953

OFFICE OF NAVAL RESEARCH

Contract Nonr-562(05)

Technical Report No. 1

Some Statistical Properties of Gated Signal Plus Narrow Band Noise

by

L. C. Maximon and J. P. Ruina

(Brown University)

DIVISION OF ENGINEERING

BROWN UNIVERSITY

PROVIDENCE, R. I.

August 1953

ABSTRACT

In this report the variance of narrow band noise plus c.w. signal after detection and gating in time is derived. The problem is solved for ideal and Gaussian filters with square law and linear detection, the latter for large predetection signal to noise ratio. In addition to these filters the analysis is carried through for a model which considers the envelope of narrow band noise plus signal as a function which is constant for intervals of duration equal to the reciprocal of the noise bandwidth and statistically independent in different intervals. A very close agreement is found among the filter types mentioned and the model. Graphs of the mean deviation divided by the mean power for noise alone are given for both filters and the model, with square law detection and predetection signal to noise ratios 0, 0.5, 1, 2. Finally we consider a simplified model which is similar to the model previously mentioned except that the gate is assumed to start at a point of change in the model voltage. The values in Table 1 show that this simplified model may be used for very large or very small gate widths but that the discrepancy is considerable for gate widths of the order of the reciprocal of the noise bandwidth.

Some Statistical Properties of Gated Signal Plus Narrow Band Noise

I. The Problem.

This paper describes some of the statistical properties of narrow band noise plus c.w. signal after detection and gating in time. More specifically, we are concerned with the fluctuation of the variables

$$y(t) = \int_{t-\delta}^{t} x(t)dt \qquad (1)$$

and

$$s(t) = \int_{t-\delta}^{t} x(t)dt - \int_{t-T-\delta}^{t-T} x(t)dt \qquad (2)$$

(see Figs. 1a and 1b)

where x(t) is the output of the system shown in Fig. 2.*

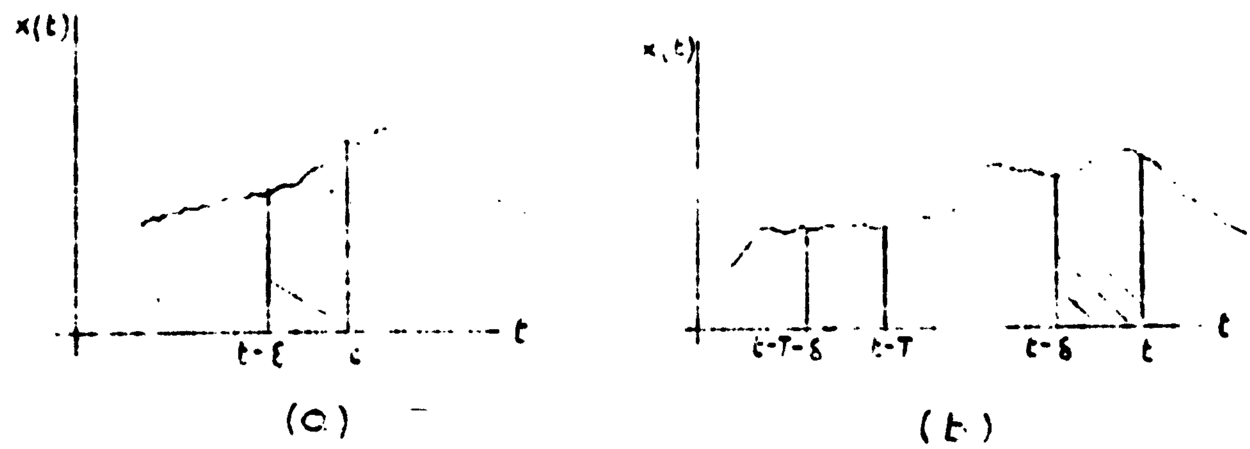


Fig. 1

*The noise bandwidth of a filter is defined as usual by $\beta = \dfrac{\int_0^\infty |Y|^2 df}{|Y(f_o)|^2}$ where Y(f) is the transfer characteristic of the filter and f_o its center frequency. Physically β represents the bandwidth of an ideal filter which has the same total power output and center frequency gain as the given filter.

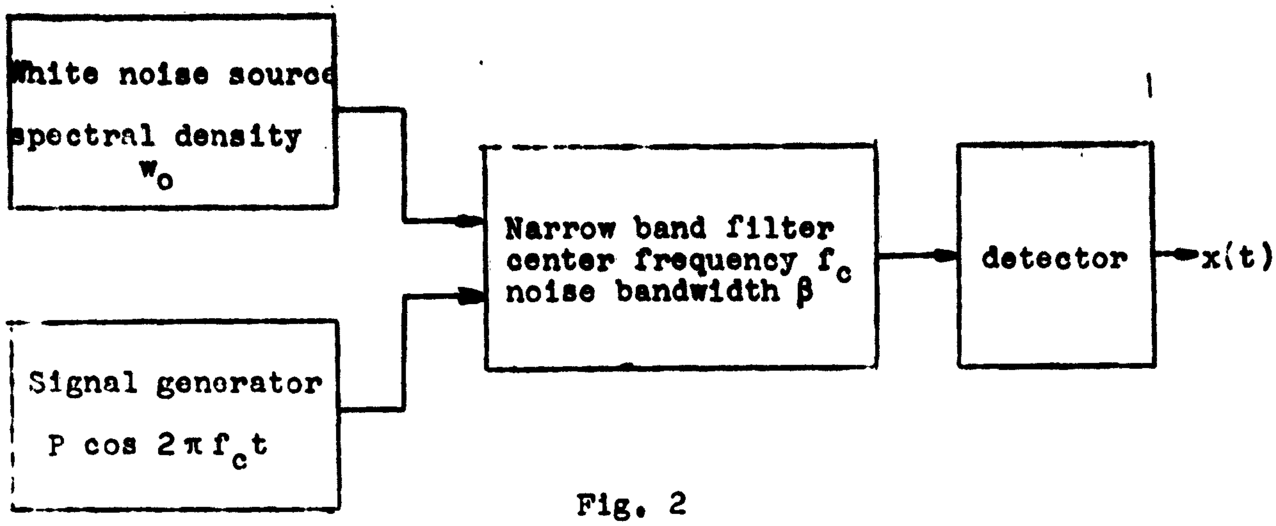


Fig. 2

The variances of y and z, which we denote by σ_y^2 and σ_z^2, will depend upon the filter and detector characteristics and are independent of time, since we shall assume $x(t)$ stationary. σ_y^2 will be a function of the parameters β, P, w_o and δ, and σ_z^2 will in addition depend on the parameter T.

We shall consider the following band pass filter characteristics:

a) Ideal Filter - transfer characteristic given by

$$Y(f) = 1 \qquad f_c - \frac{\beta}{2} < f < f_c + \frac{\beta}{2} \tag{3}$$

$$= 0 \qquad \text{elsewhere}$$

b) Gaussian Filter - transfer characteristic given by

$$Y(f) = e^{-\frac{\pi(f-f_c)^2}{2\beta^2}} \tag{4}$$

For both of these filters $\beta \ll f_c$ and the noise power output is

$$W = w_o \int_0^\infty |Y(f)|^2 df = \beta w_o$$

In addition to these filters we consider a model for the envelope of signal plus narrow band noise which is frequently used. This model considers the envelope of narrow band noise plus signal as a function which is constant for intervals of duration $\frac{1}{\beta}$ and statistically in-

dependent in different intervals. An example of such a function is shown in Fig. 3. The times at which the functions changes its value are assumed to be unknown.

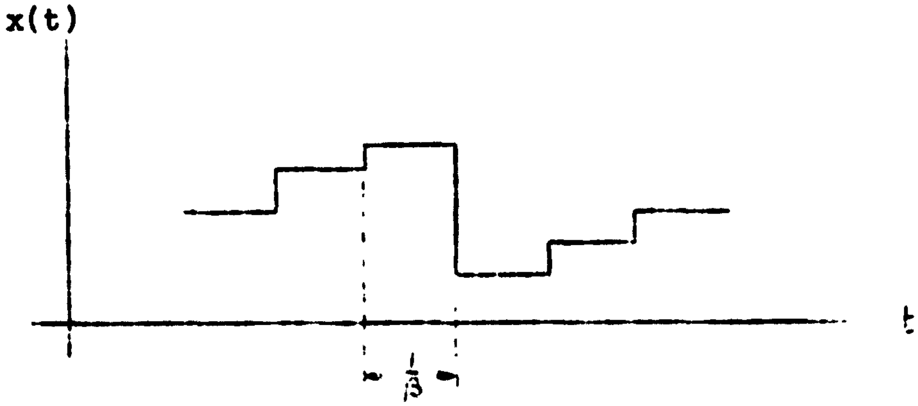


Fig. 3

The problem is solved assuming square law detection and for large signal to noise ratio in the case of the linear detector.

S. O. Rice (Jour. Acoustical Soc. Amer. 14 216 (1943), referred to hereafter as Ref. A, has solved this problem in essence for the case of zero signal. The present work extends that of Rice to include signal as well as noise. The approach is, however, different from that of Rice.

II. Noise in Linear Systems.

The gating process is a linear one and therefore the theory of noise in linear systems may be applied.

If we consider a linear system defined by a transfer characteristic $Y(f)$ or equivalently by a weighting function*

*The weighting function is the response to a unit impulse excitation.

$$h(t) = \int_{-\infty}^{\infty} Y(f)e^{2\pi ift}\, df$$

then the output $g_2(t)$ is related to the input $g_1(t)$ by the convolution integral

$$g_2(t) = \int_{-\infty}^{\infty} h(\tau)g_1(t-\tau)d\tau$$

so that

$$g_2^{\,2}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)h(\lambda)g_1(t-\tau)g_1(t-\lambda)d\tau\, d\lambda$$

and

$$\overline{g_2^{\,2}(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)h(\lambda)\,\varphi_1(\tau-\lambda)d\tau\, d\lambda \qquad (5)$$

where $\varphi_1(\tau)$ is the autocorrelation function associated with $g_1(t)$.

For the gating process described by Eqs. (1) and (2) the corresponding weighting functions are

$$h_y(t) = \begin{matrix} 1 & 0 < t < \delta \\ 0 & \text{elsewhere} \end{matrix} \qquad (6)$$

and

$$h_z(t) = \begin{matrix} 1 & 0 < t < \delta \\ -1 & T < t < T+\delta \\ 0 & \text{elsewhere} \end{matrix} \qquad (7)$$

The double integral in Eq. (5) may be reduced to a single integral by the transformation $\tau - \lambda = \eta$, $\lambda = \xi$.

For $h_y(t)$ as given in Eq. (6), we then have

$$\int_0^{\delta}\int_0^{\delta} h(\tau)h(\lambda)\,\varphi_1(\tau-\lambda)d\tau\, d\lambda = 2\int_0^{\delta} d\eta\, \varphi_1(\eta)\int_0^{\delta-\eta} h(\eta+\xi)h(\xi)d\xi$$

From Fig. 4 for $h_y(\eta+\xi)$ and $h_y(\xi)$ we have

$$\int_0^{\delta-\eta} h(\eta+\xi)h(\xi)d\xi = \delta-\eta \qquad \text{for } 0 \leqq \eta \leqq \delta$$

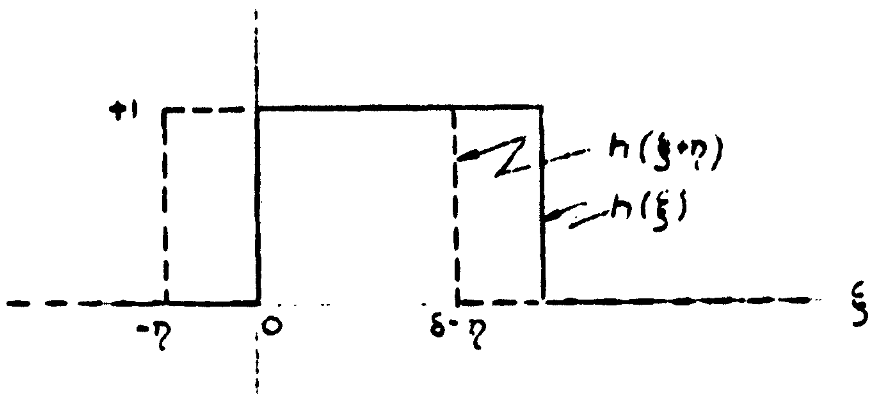


Fig. 4

so that the double integral of Eq. (5) becomes

$$\overline{\varepsilon_2^{\,2}(t)} = 2\int_0^{\delta} (\delta - \eta)\, \varphi_1(\eta)\, d\eta \tag{8}$$

(see footnote to Eq. (9))

For $h_z(t)$ as given in Eq. (7) we have

$$\int_0^{T+\delta}\int_0^{T+\delta} h(\tau)h(\lambda)\varphi_1(\tau-\lambda)\,d\tau\, d\lambda = 2\int_0^{T+\delta} d\eta\ \varphi_1(\eta)\int_0^{T+\delta-\eta} h(\eta+\xi)\, h(\xi)\, d\xi$$

From Fig. 5 for $h_z(\eta+\xi)$ and $h_z(\xi)$ one may note that there are the following contributions to the integral $\int_0^{T+\delta-\eta} h(\eta+\xi)h(\xi)d\xi$:

a) For $0 \leqq \eta \leqq \delta$ one has

$$\delta - \eta + (T + \delta - \eta) - T = 2(\delta - \eta)$$

b) For $0 \leqq T - \eta \leqq$ (i.e. $T - \delta \leqq \eta \leqq T$) one has

$$-[\delta - (T - \eta)] = [\eta - (T - \delta)]$$

c) For $0 \leqq T + \delta - \eta \leqq \delta$ (i.e. $T \leqq \eta \leqq T + \delta$) one has

$$-[T + \delta - \eta]$$

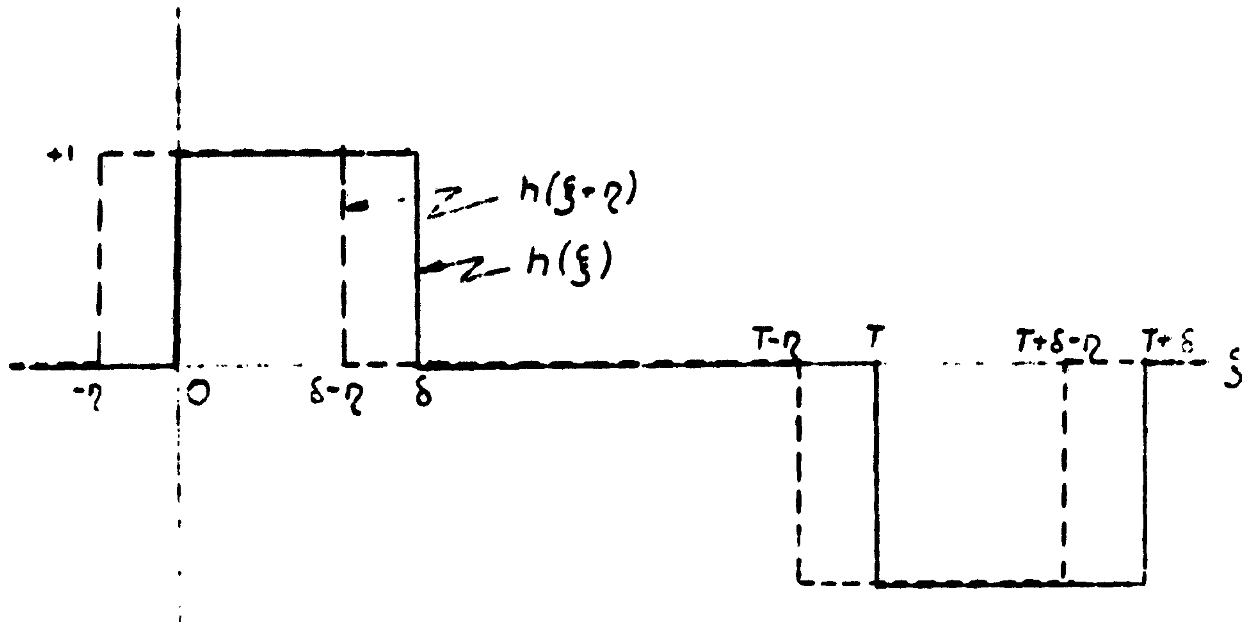


Fig. 5

Thus the double integral of Eq. (5) becomes, for the double gate:

$$\overline{g_2^{\,2}(t)} = 4\int_0^{\delta} (\delta-\eta)\varphi_1(\eta)d\eta - 2\int_{T-\delta}^{T} [\eta-(T-\delta)]\ \varphi_1(\eta)d\eta$$

$$- 2\int_T^{T+\delta} (T+\delta-\eta)\ \varphi_1(\eta)d\eta \quad * \qquad (9)$$

This result is not altered in the event that the regions mentioned in a) and b) overlap.

It may be noted that we have chosen to express $\overline{g_2^{\,2}(t)}$ as a double integral involving $h(t)$ and $\varphi(\tau)$ which is then reduced to a single integral. We could equally well have expressed $\overline{g_2^{\,2}(t)}$ directly as a single integral involving the transfer function $Y(f)$ and the input spectral density. This procedure was not followed, however, since the integrations to be performed are not nearly as simple as

---

*The variance $\sigma^2 = \overline{g_2^{\,2}(t)} - [\overline{g_2(t)}]^2$ is given by Eqs. (8) and (9). with $\varphi_1(\eta)$ replaced by $\varphi_1(\eta)$ minus its d.c. value.

those indicated in (8) and (9) for the case of the Gaussian filter and the model. We thus use (8) and (9) for the evaluation of σ_y^2 and σ_z^2 respectively (see footnote to (8) and (9)), for which we need the autocorrelation function associated with the output of the detector. This is discussed in the following section.

III. Properties of the Detector Output.

The action of the detector is such that it rectifies linearly the input signal plus narrow band noise, raises the result to some ν^{th} power and then filters so that all components of frequency of the order of the carrier frequency and above are stopped. In our analysis, however, we consider the detector as a device which first obtains the envelope of the input signal plus narrow band noise and then raises the envelope to the ν^{th} power. This output is the same as that of the actual detector apart from a constant multiplying facgor. (See Fig. 6 below.) For purposes of computation it is more convenient to consider the second description of detector action.

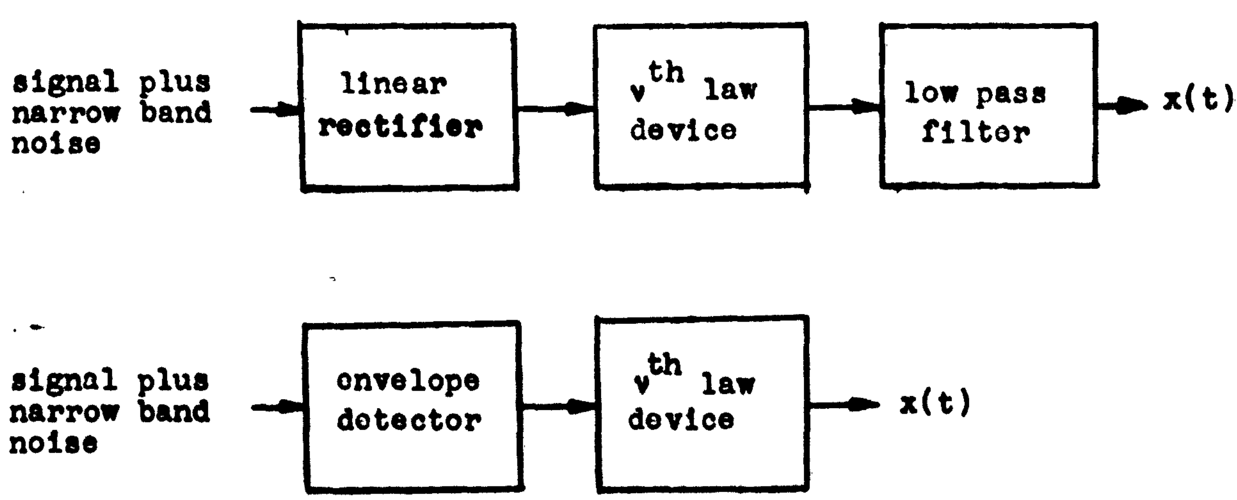


Fig. 6.

The spectrum and the equivalent autocorrelation function of the output of a square law detector have been obtained by Rice (Bell System Tech. Jour. 24, Part IV (1945)) and by Lawson and Uhlenbeck (Threshold Signals, Vol. 24 of M.I.T. Rad. Lab. Ser., McGraw-Hill, 1950. Sec. 7.2, referred to hereafter as Ref. B). The result of interest for the determination of σ_y^2 and σ_z^2 is the autocorrelation function minus its d.c. value. (See footnote following Eqs. (8) and (9)). This is given in Ref. B (p. 155, Eq. (17)) as

$$\varphi(\tau) = 4P^2W\rho(\tau) + 4W^2\rho^2(\tau) \tag{10}$$

where W = noise power input = βw_o

P = signal amplitude

and

$$\rho(\tau) = \frac{\int_0^\infty |Y|^2 \cos 2\pi(f-f_c)\tau \, df}{\int_0^\infty |Y|^2 \, df} \tag{11}$$

In the case of linear detection the autocorrelation function may also be obtained from Ref. B (p. 157, Eqs. (20a) and (20b)), but in the case of signal much greater than noise it may be obtained readily by examining the Rice representation of narrow band noise as

$$X(t) \sin 2\pi f_c t + Y(t) \cos 2\pi f_c t$$

where $X(t)$ and $Y(t)$ are slowly varying functions which have Gaussian probability distributions with variance W and have normalized autocorrelation functions = $\rho(\tau)$. Thus the envelope of narrow band noise plus signal = $P \cos 2\pi f_c t$ may be represented at any instant by the vector diagram shown in Fig. 7, and for $P^2 >> \overline{X^2} = \overline{Y^2}$ the fluctuation in R is due primarily to the fluctuation in X so that in the limit of very large P, the autocorrelation function minus its d.c. value is

$$\overline{R(t)\ R(t+\tau)} - (\overline{R})^2 = \overline{X(t)\ X(t+\tau)} = \rho W \tag{12}$$

Thus, for the ideal filter, substituting (3) in (11),

$$\rho(\tau) = \frac{\int_{f_c-\beta/2}^{f_c+\beta/2} \cos 2\pi (f-f_c)\ \tau df}{\int_{f_c-\beta/2}^{f_c+\beta/2} df}$$

$$= \frac{\sin \pi \tau \beta}{\pi\tau\beta} \tag{13}$$

and for the Gaussian filter, substituting (4) in (11)

$$\rho(\tau) = \frac{\int_0^\infty e^{-\frac{\pi(f-f_c)^2}{\beta^2}} \cos 2\pi(f-f_c)\ \tau\, df}{\int_0^\infty e^{-\frac{\pi(f-f_c)^2}{\beta^2}} df}$$

$$\approx \frac{\int_{-\infty}^{\infty} e^{-\frac{\pi f^2}{\beta^2}} \cos 2\pi f \tau df}{\int_{-\infty}^{\infty} e^{-\frac{\pi f^2}{\beta^2}}} = e^{-\pi\beta^2\tau^2} \tag{14}$$

since $\frac{f_c}{\beta} >> 1$. Eqs. (13) and (14) may then be substituted in (10) to

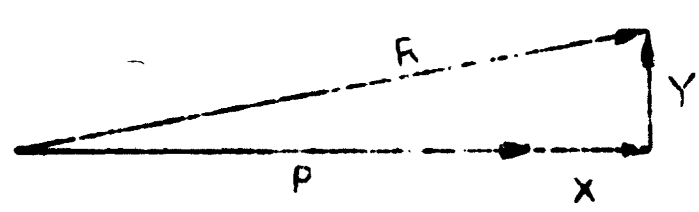


Fig. 7

obtain the autocorrelation function minus its d.c. value in the case of the ideal and Gaussian filters.

For the model the autocorrelation function is given in Fig. 8 in the case of the square law detector.

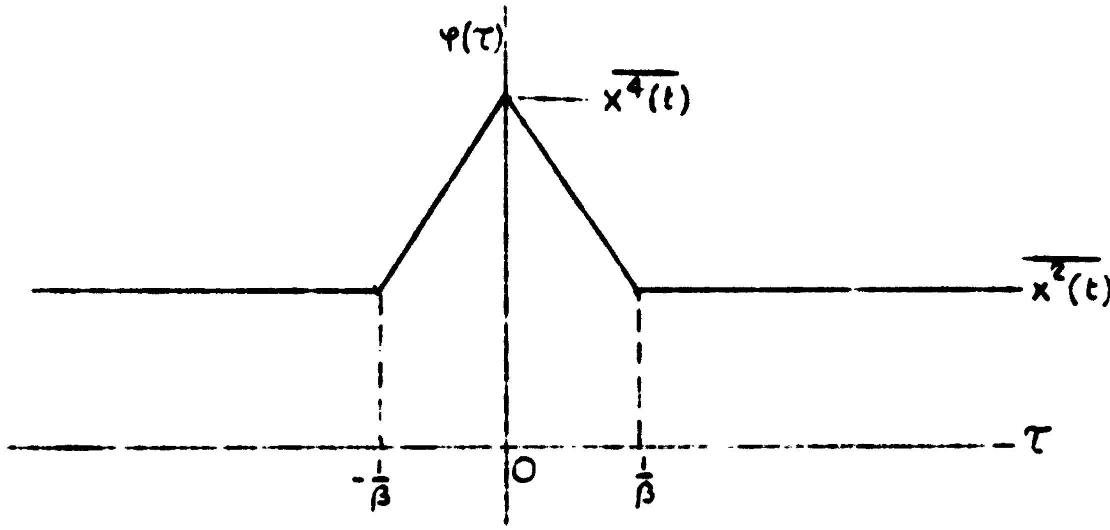


Fig. 8

From Ref. B (p. 155, Eqs. (14b) and (17)) we have

$$\overline{x^4(t)} - \overline{x^2(t)}^2 = 4WP^2 + 4W^2$$

so that the autocorrelation function minus its d.c. value is given by

$$\begin{aligned} \varphi(\tau) &= (4WP^2 + 4W^2)(1 - \beta\,|\tau|) & |\tau| \leq \frac{1}{\beta} \\ &= 0 & |\tau| > \frac{1}{\beta} \end{aligned} \qquad (15)$$

In the case of the linear detector the autocorrelation function is given in Fig. 9.

For $P^2 >> W$, $\overline{x^2(t)} - \overline{x(t)}^2$ may be obtained either from Ref. B (p. 155, Eqs. (14a) and (14b), retaining the first two terms in the asymptotic expansion of (14a)) or directly by setting $\tau = 0$ in (12), giving

$$\overline{x^2(t)} - \overline{x(t)}^2 = W$$

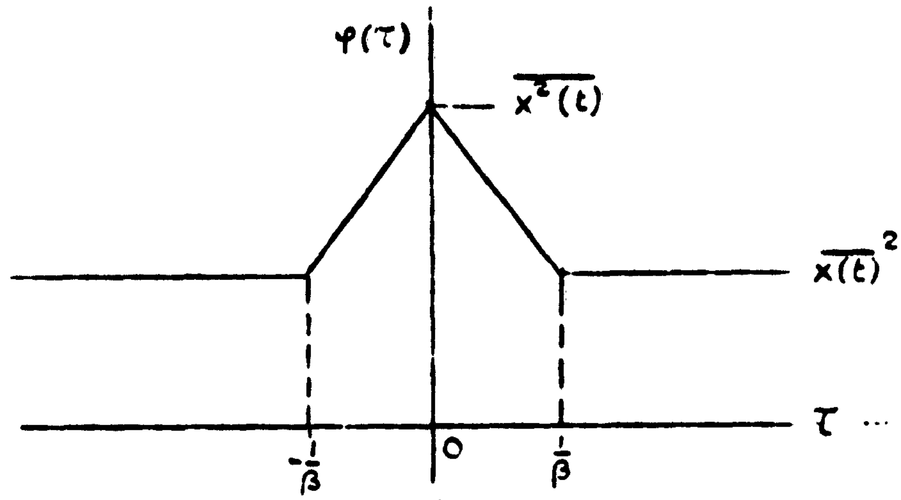


Fig. 9

so that the autocorrelation function minus its d.c. value is

$$\varphi(\tau) = W(1 - \beta\,|\tau|) \qquad |\tau| \leq \frac{1}{\beta}$$
$$= 0 \qquad |\tau| > \frac{1}{\beta} \tag{16}$$

IV. Results, Discussion and Conclusions.

σ_y^2 and σ_z^2 can now be determined for the ideal and Gaussian filters from (13), (14), (10) and (12) by substituting in (8) and (9), and for the model by substituting (15) and (16) in (8) and (9). The integrations are carried out in the appendix and the results are as follows:

For the single gate, square law detector

Ideal filter $\quad \sigma_y^2 = \dfrac{4W^2}{(\pi\beta)^2}\left[G(2\pi\beta\delta) + 4\mu\pi\beta\delta\, G'(\pi\beta\delta)\right]^{*}$ (17)

---

*To perform the integrations it was found convenient to introduce the function

$$G(x) = \int_0^x ds \int_0^s \frac{1-\cos t}{t^2}\,dt$$

defined by

$$G''(x) = \frac{1-\cos x}{x^2}, \quad G'(o) = G(o) = 0 \;.$$

The values of $G(x)$ and $G'(x)$ for $0 \leq x \leq 100$ are given in Table 2 at the end of this report.

Gaussian filter

$$\sigma_y^2 = \frac{W^2}{\beta^2\sqrt{\pi}}\left[2\beta\delta\ \sqrt{2\pi}\ \mathrm{Erf}(\beta\delta\ \sqrt{2\pi}) + \frac{2}{\sqrt{\pi}}\, e^{-2\pi\beta^2\delta^2} - \frac{2}{\sqrt{\pi}}\right] \tag{18}$$

$$\text{Model}\quad \sigma_y^2 = 4W^2\delta^2(1 + 2\mu)\left(1 - \frac{\beta\delta}{3}\right),\quad \beta\delta \leq 1$$

$$= 4W^2\delta^2(1 + 2\mu)\,\frac{(\beta\delta - \frac{1}{3})}{(\beta\delta)^2},\quad \beta\delta \geq 1 \tag{19}$$

where $\mu = \frac{P^2}{2W}$ = predetection signal to noise ratio.

For $\beta\delta \ll 1$, σ_y^2 for both filters as well as the model should be the variance of a single sample, of width δ, i.e. we should expect to have

$$\sigma_y^2 = \delta^2\left[\overline{x^4(t)} - \overline{x^2(t)}^2\right] = \delta^2[4WP^2 + 4W^2] = 4W^2\delta^2(1 + 2\mu) \tag{20}$$

(see Ref. B., p. 155, Eqs. (14b) and (17)). Substituting the expressions for G(x) and Erf(x) as given in the appendix (Eqs. (A-11)), (17), (18) and (19) reduce to (20) in the limit for $\beta\delta \ll 1$.

For $\beta\delta \gg 1$, we may use the asymptotic expressions for G(x) and Erf(x) given in the appendix, Eqs. (A-12), and obtain the following limits from (17, (18) and (19):

$$\text{Ideal filter}\qquad \sigma_y^2 = \frac{4W^2}{\beta^2}(1 + 2\mu)\beta\delta \tag{21}$$

$$\text{Gaussian filter}\qquad \sigma_y^2 = \frac{4W^2}{\beta^2}\left(\frac{\sqrt{2}}{2} + 2\mu\right)\beta\delta \tag{22}$$

$$\text{Model}\qquad \sigma_y^2 = \frac{4W^2}{\beta^2}(1 + 2\mu)\ \beta\delta \tag{23}$$

In each of these cases σ_y^2 is proportional to δ as is to be expected since for $\beta\delta \gg 1$ the gating process is effectively adding up a number (of order δ) of independent random variables, the variance of the sum then being the sum of the variances. It is to be noted that

both for $\beta\delta << 1$ and $\beta\delta >> 1$ the σ_y^2 corresponding to the ideal filter and the model are equal for all signal to noise ratios. The values of σ_y^2 for the ideal filter and model do not differ appreciably even for $\beta\delta \sim 1$, as is shown by the figures given in Table 1.

For the particular case of zero signal the expressions given in (17) and (18) for σ_y^2 for the ideal and Gaussian filters with square law detector and single gate are compared with the corresponding expressions given in Ref. A (p. 223, Eq. (6.10) and p. 225) in the appendix, and are shown to be four times the variances given in Ref. A. This is to be expected since in Ref. A Rice considers the short time average value of the noise after detection, whereas we have been considering the envelope, and these differ by a factor of two.

For the single gate and linear detector with large signal to noise ratio, we have the following results:

Ideal filter $$\sigma_y^2 = 2W\delta^2 \frac{G'(\pi\beta\delta)}{\pi\beta\delta} \tag{24}$$

Gaussian filter

$$\sigma_y^2 = \frac{W}{2\beta^2\sqrt{\pi}}\left[2\beta\delta\sqrt{\pi} \quad \mathrm{Erf}\,(\beta\delta\sqrt{\pi}) + \left(\frac{2}{\sqrt{\pi}}\, e^{-\pi\beta^2\delta^2} - \frac{2}{\sqrt{\pi}}\right)\right] \tag{25}$$

Model $$\sigma_y^2 = W\delta^2\left(1 - \frac{\beta\delta}{3}\right), \qquad \beta\delta \leq 1$$

$$= W\delta^2 \frac{(\beta\delta - 1/3)}{(\beta\delta)^2} \qquad \beta\delta \geq 1 \tag{26}$$

Again, for $\beta\delta << 1$, σ_y^2 for both filters as well as the model should be the variance of a single sample, of width δ, i.e. we should expect to have

$$\sigma_y^2 = \delta^2\left[\overline{x^2(t)} - \overline{x(t)}^2\right] = \delta^2 W \tag{27}$$

Substituting the expressions for G(x) and Erf(x) as given in the ap-

pendix (Eqs. (A-11)), (24), (25) and (26) each reduce to (27) in the limit for $\beta\delta \ll 1$.

For $\beta\delta \gg 1$, we may use the asymptotic expressions for $G(x)$ and $Erf(x)$ given in the appendix, Eqs. '12). We then obtain, from (24), (25) and (26),

$$\sigma_y^2 = \frac{W}{\beta^2} \cdot \beta\delta \tag{27a}$$

for both the ideal and Gaussian filters and the model. In (27a), as in (21), (22) and (23), σ_y^2 is proportional to δ, as is to be expected.

For the double gate with square law detector we have the following results:

Ideal filter

$$\sigma_z^2 = \frac{8W^2}{(\pi\beta)^2}\begin{bmatrix} G(2\pi\beta\delta) + 4\mu\pi\beta\delta \; G'(\pi\beta\delta) \\ + G(2\pi\beta T) + 4\mu\pi\beta T \; G'(\pi\beta T) \end{bmatrix} - \frac{4W^2}{(\pi\beta)^2}\begin{bmatrix} G(2\pi\beta(T-\delta)) + 4\mu\pi\beta(T-\delta)G'(\pi\beta(T-\delta)) \\ + G(2\pi\beta(T+\delta)) + 4\mu\pi\beta(T+\delta)G'(\pi\beta(T+\delta)) \end{bmatrix} \tag{28}$$

Gaussian filter

$$\sigma_z^2 = \frac{2W^2}{\beta^2\sqrt{\pi}}\left[\begin{aligned} & 2\beta\delta\sqrt{2\pi}\,Erf(\beta\delta\sqrt{2\pi}) + 2\beta T\sqrt{2\pi}\,Erf(\beta T\sqrt{2\pi}) \\ & -\beta(T-\delta)\sqrt{2\pi}\,Erf(\beta(T-\delta)\sqrt{2\pi}) - \beta \\ & \qquad - \beta(T+\delta)\sqrt{2\pi}\,Erf(\beta(T+\delta)\sqrt{2\pi}) \\ & + \frac{2}{\sqrt{\pi}}e^{-2\pi\beta^2\delta^2} - \frac{2}{\sqrt{\pi}} \\ & - \frac{1}{2}\left[\frac{2}{\sqrt{\pi}}e^{-2\pi\beta^2(T-\delta)^2} - 2\frac{2}{\sqrt{\pi}}e^{-2\pi\beta^2T^2} + \frac{2}{\sqrt{\pi}}e^{-2\pi\beta^2(T+\delta)^2}\right] \end{aligned}\right] \tag{29}$$

Gaussian filter (cont'd)

$$+\frac{8\mu W^2}{\beta^2\sqrt{\pi}}\left[\begin{array}{l} 2\beta\delta\sqrt{\pi}\ \mathrm{Erf}(\beta\delta\sqrt{\pi}) + 2\beta T\sqrt{\pi}\ \mathrm{Erf}(\beta T\sqrt{\pi}) \\ -\beta(T-\delta)\sqrt{\pi}\ \mathrm{Erf}(\beta(T-\delta)\sqrt{\pi}) - \beta(T+\delta)\sqrt{\pi}\ \mathrm{Erf}(\beta(T+\delta)\sqrt{\pi}) \\ +\frac{2}{\sqrt{\pi}} e^{-\pi\beta^2\delta^2} - \frac{2}{\sqrt{\pi}} \\ -\frac{1}{2}\left[\frac{2}{\sqrt{\pi}} e^{-\pi\beta^2(T-\delta)^2} - 2\frac{2}{\sqrt{\pi}} e^{-\pi\beta^2T^2} + \right. \\ \qquad \left. + \frac{2}{\sqrt{\pi}} e^{-\pi\beta^2(T+\delta)^2}\right] \end{array}\right]$$

Model

a) $$\sigma_z^2 = \frac{8W^2}{\beta^2}(1+2\mu)(\beta\delta)^2\left[\beta T - \frac{1}{3}\beta\delta\right] \qquad \beta(T+\delta) < 1$$

b) $$\sigma_z^2 = \frac{8W^2}{\beta^2}(1+2\mu)\left[(\beta\delta)^2 - \frac{1}{3}(\beta\delta)^3 - \frac{\beta^2(T-\delta)^2}{2} + \frac{\beta^3(T-\delta)^3}{6} - \frac{\beta(T+\delta)}{2} + \frac{1}{6} + (\beta T)^2 - \frac{1}{3}(\beta T)^3\right]$$

$$\beta T < 1 \leq \beta(T+\delta) \qquad (30)$$

c) $$\sigma_z^2 = \frac{8W^2}{\beta^2}(1+2\mu)\left[(\beta\delta)^2 - \frac{1}{3}(\beta\delta)^3 - \frac{1}{6} - \frac{\beta^2(T-\delta)^2}{2} + \frac{\beta^3(T-\delta)^3}{2} + \beta\frac{(T-\delta)}{2}\right]$$

$$\beta\delta < 1$$

$$\beta(T-\delta) < 1 \leq \beta T$$

d) $$\sigma_z^2 = \frac{8W^2}{\beta^2}(1+2\mu)\left[(\beta\delta)^2 - \frac{1}{3}(\beta\delta)^3\right]$$

$$\beta\delta < 1$$

$$1 \leq \beta(T-\delta)$$

e) $\sigma_z^{\,2} = \frac{8W^2}{\beta^2}(1+2\mu)\left[\beta\delta - \frac{1}{2} + \beta\frac{(T-\delta)}{2} + \beta^2\frac{(T-\delta)^2}{2} + \beta^3\frac{(T-\delta)^3}{6}\right]$

$1 \leqq \beta\delta$

$\beta(T-\delta) < 1 \leqq \beta T$

f) $\sigma_z^{\,2} = \frac{8W^2}{\beta^2}(1+2\mu)(\beta\delta - \frac{1}{3})$ $\qquad 1 \leqq \beta\delta$

$1 \leqq \beta(T-\delta)$

For $\beta\delta << 1$, $\sigma_y^{\,2}$ should be the variance of the difference of two samples, each of width δ, separated by a distance T, i.e. we should expect to have

$$\sigma_z^{\,2} = \delta^2\left[\overline{(x^2(t) - x^2(t+T))^2} - \overline{(x^2(t) - x^2(t+T))}^{\,2}\right]$$

$$= 2\delta^2[\overline{x^4(t)} - \overline{x^2(t)x^2(t+T)}] \tag{31}$$

$$= 8W^2\delta^2[(1-\rho^2(T)) + 2\mu(1-\rho(T))]$$

(See Ref. B, p. 155, Eq. (17))

For the ideal filter this is

$$\sigma_z^{\,2} = 8W^2\delta^2\left[1 - \frac{\sin^2\pi\beta T}{(\pi\beta T)^2} + 2\mu\left(1 - \frac{\sin\pi\beta T}{\pi\beta T}\right)\right] \tag{31a}$$

and for the Gaussian filter

$$\sigma_z^{\,2} = 8W^2\delta^2[1 - e^{-2\pi\beta^2T^2} + 2\mu(1 - e^{-\pi\beta^2T^2})] \tag{31b}$$

(31a) and (31b) may be obtained directly from (28) and (29) by examining these expressions in the limit of $\beta\delta << 1$.

In the case of the model we have, for $\beta\delta << 1$

$$\sigma_z^{\,2} = 8W^2\delta^2(1+2\mu)\beta T \qquad \text{for } \beta T < 1, \text{ from (30a)} \tag{31c}$$

$$\sigma_z^{\,2} = 8W^2\delta^2(1+2\mu) \qquad \text{for } \beta T > 1, \text{ from (30d)} \tag{31d}$$

For $\beta\delta << 1$, $\beta T >> 1$ we have, for both filters as well as for

the model,

$$\sigma_z^{\,2} = 8W^2\,\delta^2(1 + 2\,\mu) \tag{32}$$

which is twice the variance of a single sample of width δ (see Eq. 20), as is to be expected.

For $\beta\delta >> 1$, we have, substituting the asymptotic expressions for $G(x)$, $G'(x)$ and $Erf(x)$ in (28) and (29), and directly from (30),

$$\sigma_z^{\,2} = \frac{8W^2}{\beta^2}(1 + 2\mu)\,\beta\delta \tag{33}$$

for the ideal filter and the model, and

$$\sigma_z^{\,2} = \frac{8W^2}{\beta^2}\left(\frac{\sqrt{2}}{2} + 2\,\mu\right)\beta\delta \tag{34}$$

$\sigma_z^{\,2}$ as given in (33) and (34) are twice the variance for a single gate for which $\beta\delta >> 1$ (see Eqs. (21), (22) and (23)), as should be expected. It may be noted that (33) and (34) are obtained for $\beta\delta >> 1$ regardless of whether $\beta(T-\delta) << 1$, $\beta(T-\delta) \sim 1$ or $\beta(T-\delta) >> 1$.

For the particular case of zero signal the expression given in (27) for $\sigma_y^{\,2}$ for the ideal filter with square law detector and double gate is compared with the corresponding expression given in Ref. A (-. 224, Eq. (7.3)) in the appendix, and is four times the variance given in Ref. A, as it should be, since in Ref. A the average value of the noise rather than its envolope is considered.

For the double gate and linear detector with large signal to noise ratio, the results are as follows:

Ideal filter:

$$\sigma_z^{\,2} = \frac{4W}{(\pi\,\beta)^2}\left[\pi\beta\delta\, G'(\pi\beta\delta) + \pi\,\beta\, T G'(\pi\,\beta\, T)\right]$$
$$- \frac{2W}{(\pi\beta)^2}\left[\pi\,\beta\,(T-\delta)G'(\pi\,\beta\,(T-\delta)) + \pi\beta(T+\delta)G'(\pi\beta(T+\delta))\right] \tag{35}$$

Gaussian filter:

$$\sigma_z^{\,2} = \frac{W}{\beta^2\sqrt{\pi}}\left[\begin{array}{l} 2\,\beta\delta\sqrt{\pi}\ \mathrm{Erf}(\beta\delta\sqrt{\pi}) + 2\,\beta T\sqrt{\pi}\ \mathrm{Erf}(\beta T\sqrt{\pi}) \\ -\,\beta(T-\delta)\sqrt{\pi}\ \mathrm{Erf}(\beta(T-\delta)\sqrt{\pi}) - \\ -\,\beta(T+\delta)\sqrt{\pi}\ \mathrm{Erf}(\beta(T+\delta)\sqrt{\pi}) \\ +\,\frac{2}{\sqrt{\pi}}\,e^{-\pi\beta^2\delta^2} - \frac{2}{\sqrt{\pi}} \\ -\,\frac{1}{2}\left[\frac{2}{\sqrt{\pi}}\,e^{-\pi\beta^2(T-\delta)^2} - \right. \\ \left. -\,2\,\frac{2}{\sqrt{\pi}}\,e^{-\pi\beta^2T^2} + \frac{2}{\sqrt{\pi}}\,e^{-\pi\beta^2(T+\delta)^2}\right] \end{array}\right] \qquad (36)$$

Model:

a) $$\sigma_z^{\,2} = \frac{2W}{\beta^2}(\beta\delta)^2\left[\beta T - \frac{1}{3}\beta\delta\right] \qquad \beta(T+\delta) < 1$$

b) $$\sigma_z^{\,2} = \frac{2W}{\beta^2}\left[\begin{array}{l}(\beta\delta)^2 - \frac{1}{3}(\beta\delta)^3 - \beta^2\frac{(T-\delta)^2}{2} + \beta^3\frac{(T-\delta)^3}{6} \\ -\,\beta\frac{(T+\delta)}{2} + \frac{1}{6} + (\beta T)^2 - \frac{1}{3}(\beta T)^3\end{array}\right]$$

$$\beta T < 1 \leq \beta(T+\delta)$$

c) $$\sigma_z^{\,2} = \frac{2W}{\beta^2}\left[\begin{array}{l}(\beta\delta)^2 - \frac{1}{3}(\beta\delta)^3 - \frac{1}{6} - \beta^2(T-\delta)^2 \quad \beta^3(T-\delta)^3 \\ +\,\beta\frac{(T-\delta)}{2}\end{array}\right] \qquad (37)$$

$$\beta\delta < 1$$

$$\beta(T-\delta) < 1 \leq \beta T$$

d) $$\sigma_z^{\,2} = \frac{2W}{\beta^2}\left[(\beta\delta)^2 - \frac{1}{3}(\beta\delta)^3\right] \qquad \beta\delta < 1$$

$$1 \leq \beta(T-\delta)$$

e) $$\sigma_z^{\,2} = \frac{2W}{\beta^2}\left[\beta\delta - \frac{1}{2} + \beta\frac{(T-\delta)}{2} + \beta^2\frac{(T-\delta)^2}{2} + \beta^3\frac{(T-\delta)^3}{6}\right]$$

$$1 \leq \beta\delta$$

$$\beta(T-\delta) < 1 \leq \beta T$$

f) $$\sigma_z^{\,2} = \frac{2W}{\beta^2}\left(\beta\delta - \frac{1}{3}\right) \qquad 1 \leq \beta\delta$$

$$1 \leq \beta(T-\delta)$$

Again, for $\beta\delta << 1$, σ_y^2 should be the variance of the difference of two samples, each of width δ, separated by a distance T, i.e. we should expect to have

$$\sigma_z^2 = \delta^2 \left[\overline{(x(t) - x(t+T))^2} - \overline{(x(t) - x(t+T))}^2\right]$$

$$= 2\delta^2 \left[\overline{x^2(t)} - \overline{x(t)x(t+T)}\right]$$

$$= 2\delta^2 W(1 - \rho(T))\text{\#} \tag{38}$$

for large signal to noise ratio.

For the ideal filter this is

$$\sigma_z^2 = 2W\delta^2\left(1 - \frac{\sin \pi\beta T}{\pi\beta T}\right) \tag{38a}$$

and for the Gaussian filter

$$\sigma_z^2 = 2W\delta^2\left(1 - e^{-\pi\beta^2 T^2}\right) \tag{38b}$$

(38a) and (38b) may be obtained directly from (35) and (36) by examining these expressions in the limit of $\beta\delta << 1$.

In the case of the model, we have, for $\beta\delta << 1$,

$$\sigma_z^2 = 2W\delta^2 \cdot \beta T \qquad \text{for } \beta T < 1 \text{, from (37a)} \tag{38c}$$

$$\sigma_z^2 = 2W\delta^2 \qquad \text{for } \beta T > 1 \text{, from (37d)}$$

For $\beta\delta >>$, $\beta T >> 1$, we have, for both filters as well as for the model

---

\#From Eq. (12), for large signal to noise ratio, we have $\overline{x(t)x(t+T)} - \overline{x(t)}^2 = \rho(T)W$, and, setting T = 0, $\overline{x^2(t)} - \overline{x(t)}^2 = W$.

Thus $\overline{x^2(t)} - \overline{x(t)x(t+T)} = W(1- \rho(T))$

$$\sigma_z^{\,2} = 2W\,\delta^2 \tag{39}$$

which is twice the variance of a single sample of width δ (see Eq. (27).

For $\beta\delta >> 1$ we have, substituting the asymptotic expressions for $G(x)$, $G'(x)$ and $Erf(x)$ in (35) and (36) and directly from (37),

$$\sigma_z^{\,2} = \frac{2W}{\beta^2}\,\beta\delta \tag{40}$$

for both filters as well as for the model. The remarks made following (34) apply equally well to (40), (cf. Eq. (27a).

In the accompanying graphs (Figs. 10, 11, 12), the following plots have been drawn, in each case for the square law detector only:

1) $\frac{\sigma_Y}{m}$ as a function of $\beta\delta$ for both filters and the model, and in each case for pre-detection signal to noise ratio $\mu = \frac{P^2}{2W} = 0$, 0.5, 1.0 and 2.0, where $m = 2W\delta$ is the mean power for noise alone and gate width δ. This same quantity has been plotted in Ref. A (p. 217, Fig. 1) for the ideal and Gaussian filters in the case of no signal, and is denoted there by $\frac{\sigma_T}{m_T}$.

2) $\frac{\sigma_z}{m}$ as a function of $\beta\delta$, with $T = \delta$, for both filters and the model, and in each case for signal to noise ratio $\mu = \frac{P^2}{2W} = 0$, 0.5, 1.0 and 2.0, where $m = 2W\delta$. For the particular case of the ideal filter and no signal this plot is given in Ref. A (p. 217, Fig. 1) and is denoted there by $\frac{\sigma_{T,T}}{m_T}$.

3) $\frac{\sigma_z}{m}$ as a function of βT, in the limit as $\delta \rightarrow 0$, for both filters and the model, with m and the values of μ as given in above. These graphs show a very good agreement among the filter types chosen and the model. For most applications use of the model is probably justified. It should be noted, however, that when the model is used in a very simple manner for computations, that is, by assuming that

the gate starts at a point of change in the model voltage, then the results are not as good, as is shown by the values of $\frac{\sigma_Y}{m}$ given in Table 1 for $\beta\delta = 0, 1, 2, 3, 4, 5$.

Applying the model to the gating process is essentially the same as adding a certain number of independent variables. When the model is used in the simple manner described above, the number of independent variables added is necessarily less than or equal to the number added when the model is used as in this paper. For equal gate durations, both methods will give the same mean value, but the simple method will result in a larger variance.* For $\beta\delta >> 1$ and for $\beta\delta << 1$, both methods agree and the greatest difference is when $\beta\delta \sim 1$.

*As a simple example of this principle, consider two independent random variables x and y having the same variance σ^2. If, however, we compute the variance of $\frac{x+y}{2}$ we get $\frac{\sigma^2}{2}$.

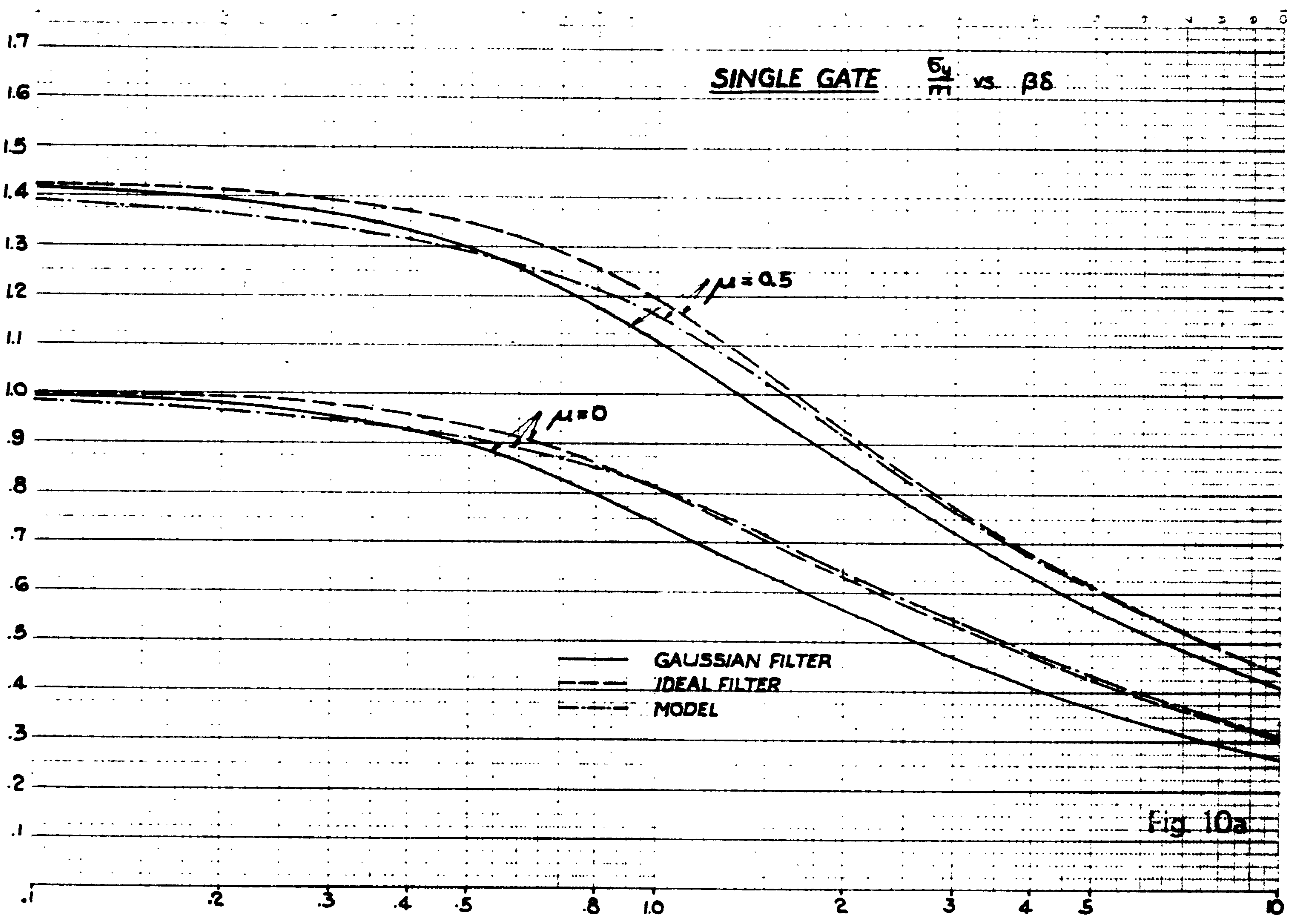
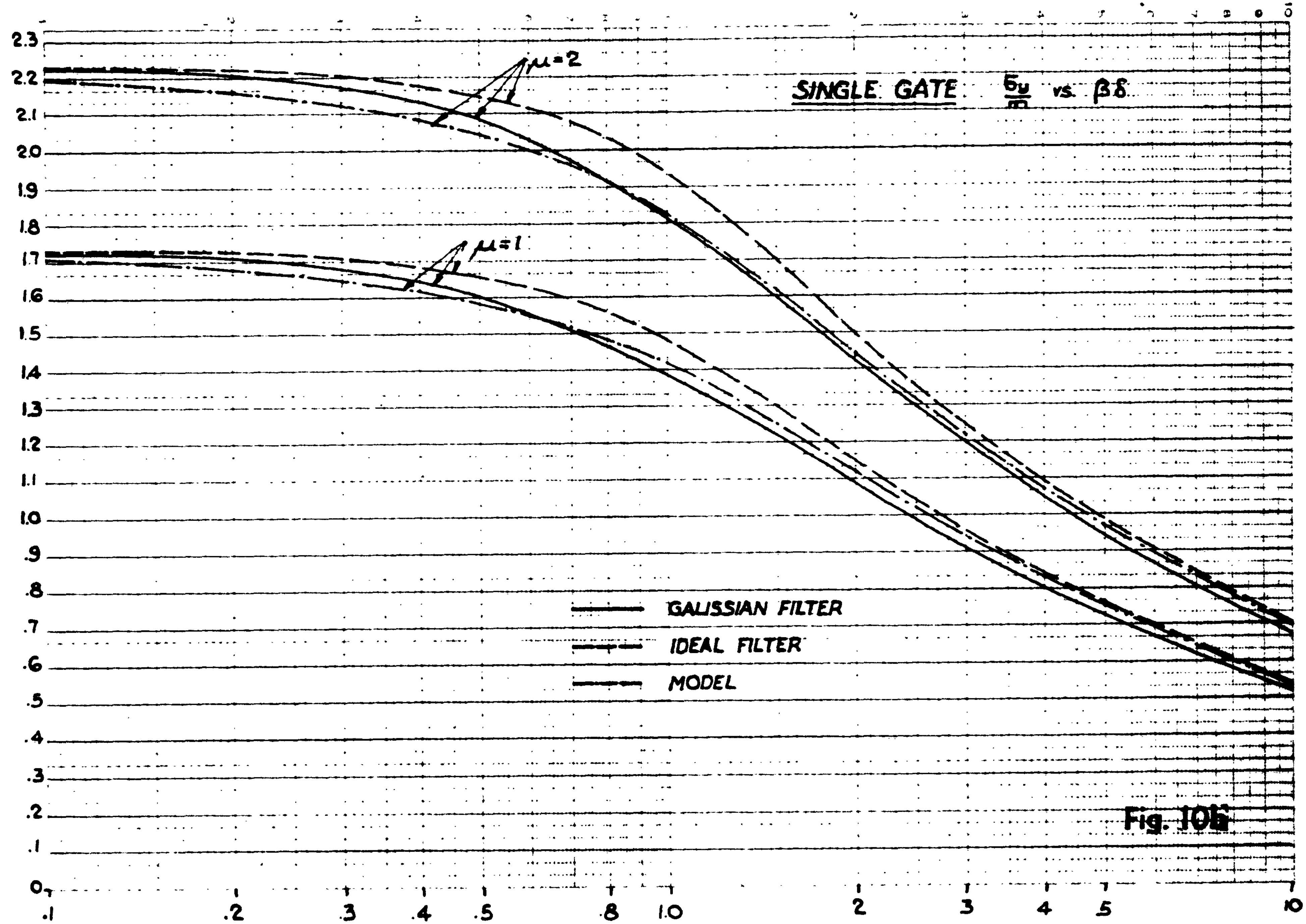

SINGLE GATE $\frac{\sigma_y}{m}$ vs. $\beta\delta$
$\mu = 0.5$
$\mu = 0$
GAUSSIAN FILTER
IDEAL FILTER
MODEL
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
.1
.2
.3
.4
.5
.8
1.0
2
3
4
5
10

Fig. 10a



SINGLE GATE $\frac{\sigma_y}{m}$ vs. $\beta\delta$
$\mu=2$
$\mu=1$
GAUSSIAN FILTER
IDEAL FILTER
MODEL
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.8
1.0
2
3
4
5
10

Fig. 10b

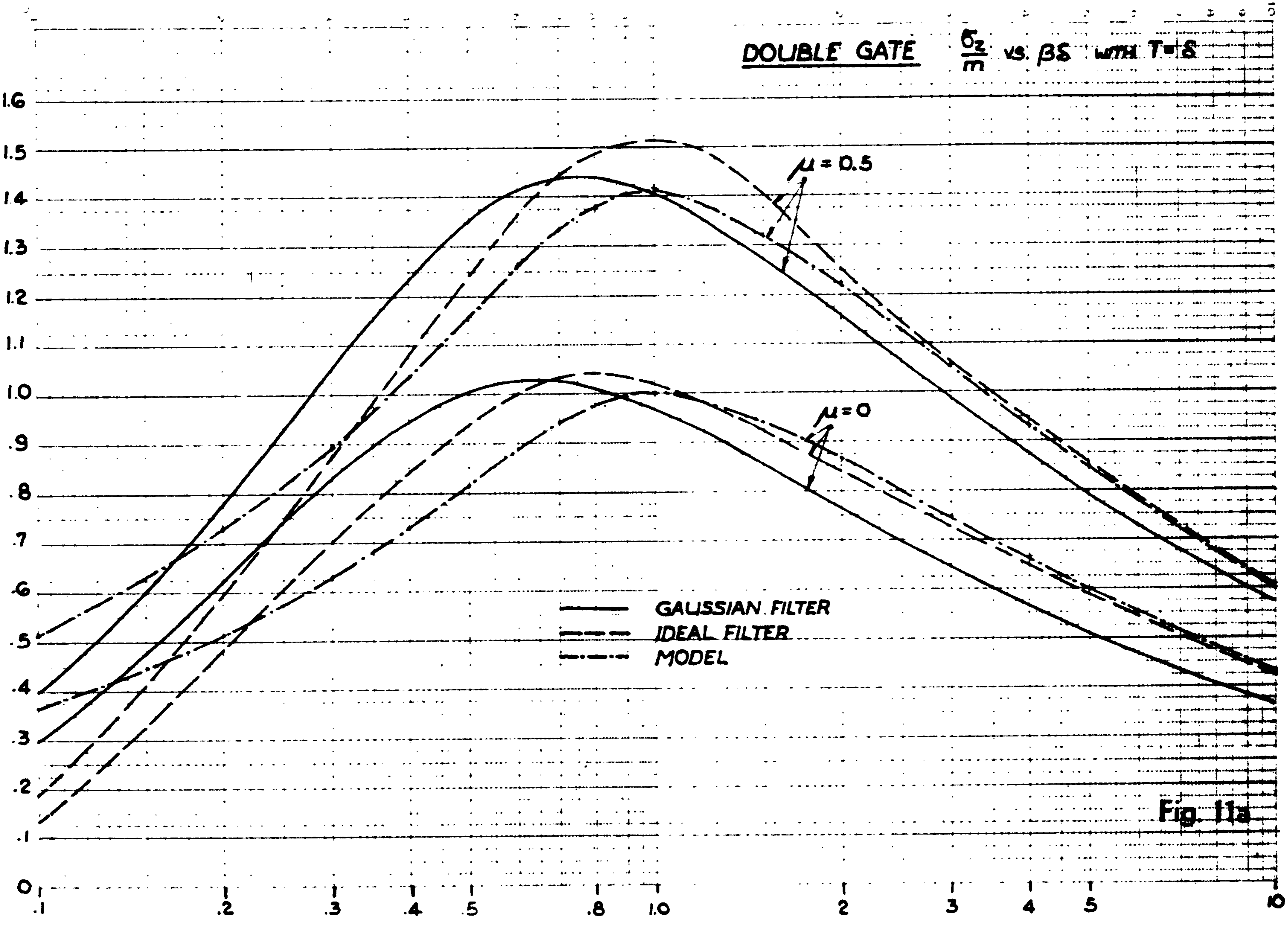

DOUBLE GATE $\frac{\sigma_z}{m}$ vs. βS with $T=S$
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.8
1.0
2
3
4
5
10
$\mu = 0.5$
$\mu = 0$
GAUSSIAN FILTER
IDEAL FILTER
MODEL

Fig. 11a

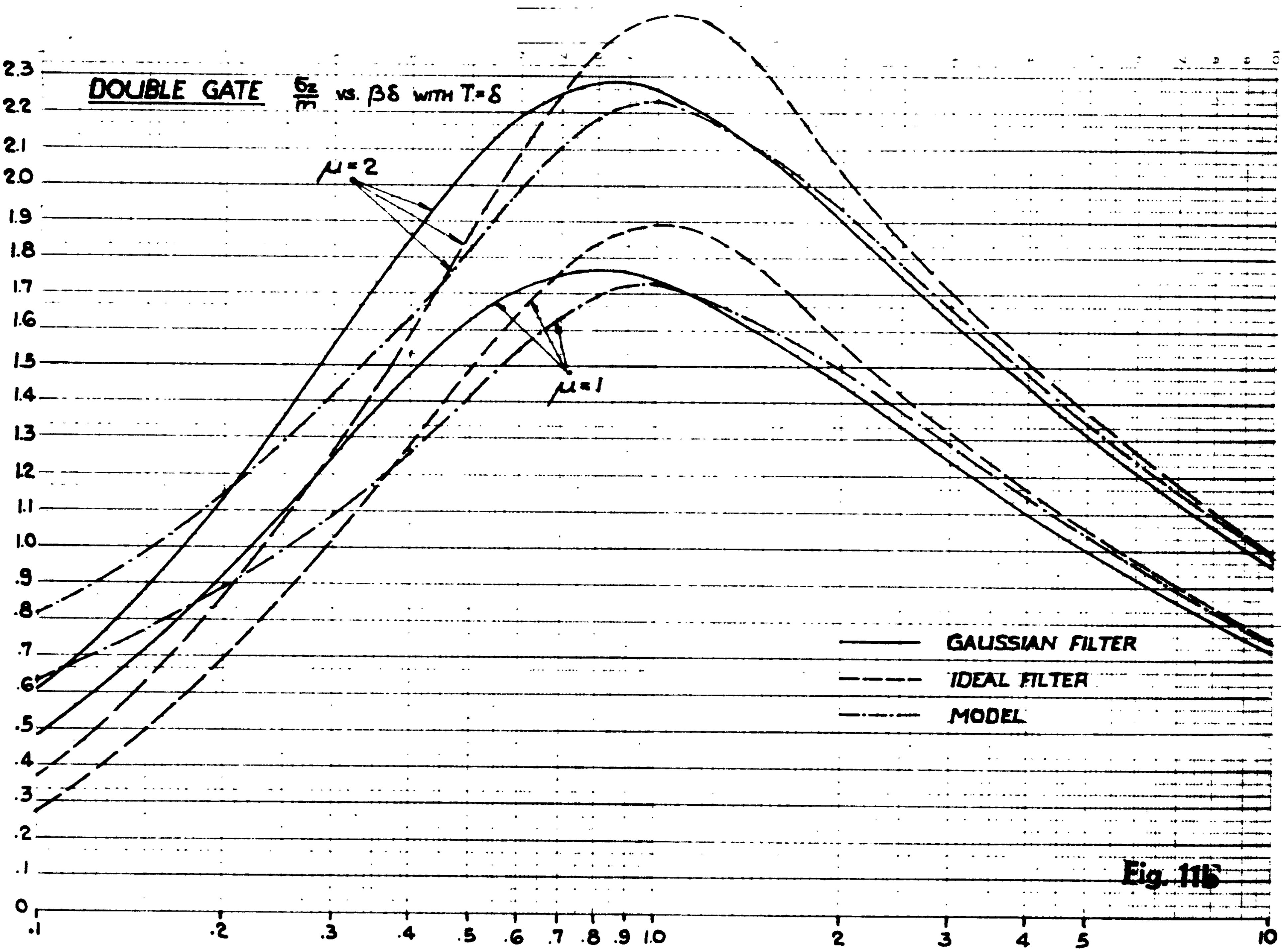

DOUBLE GATE $\frac{\sigma_z}{m}$ vs. $\beta\delta$ with $T=\delta$
$\mu=2$
$\mu=1$
GAUSSIAN FILTER
IDEAL FILTER
MODEL
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
2
3
4
5
10

Fig. 11b

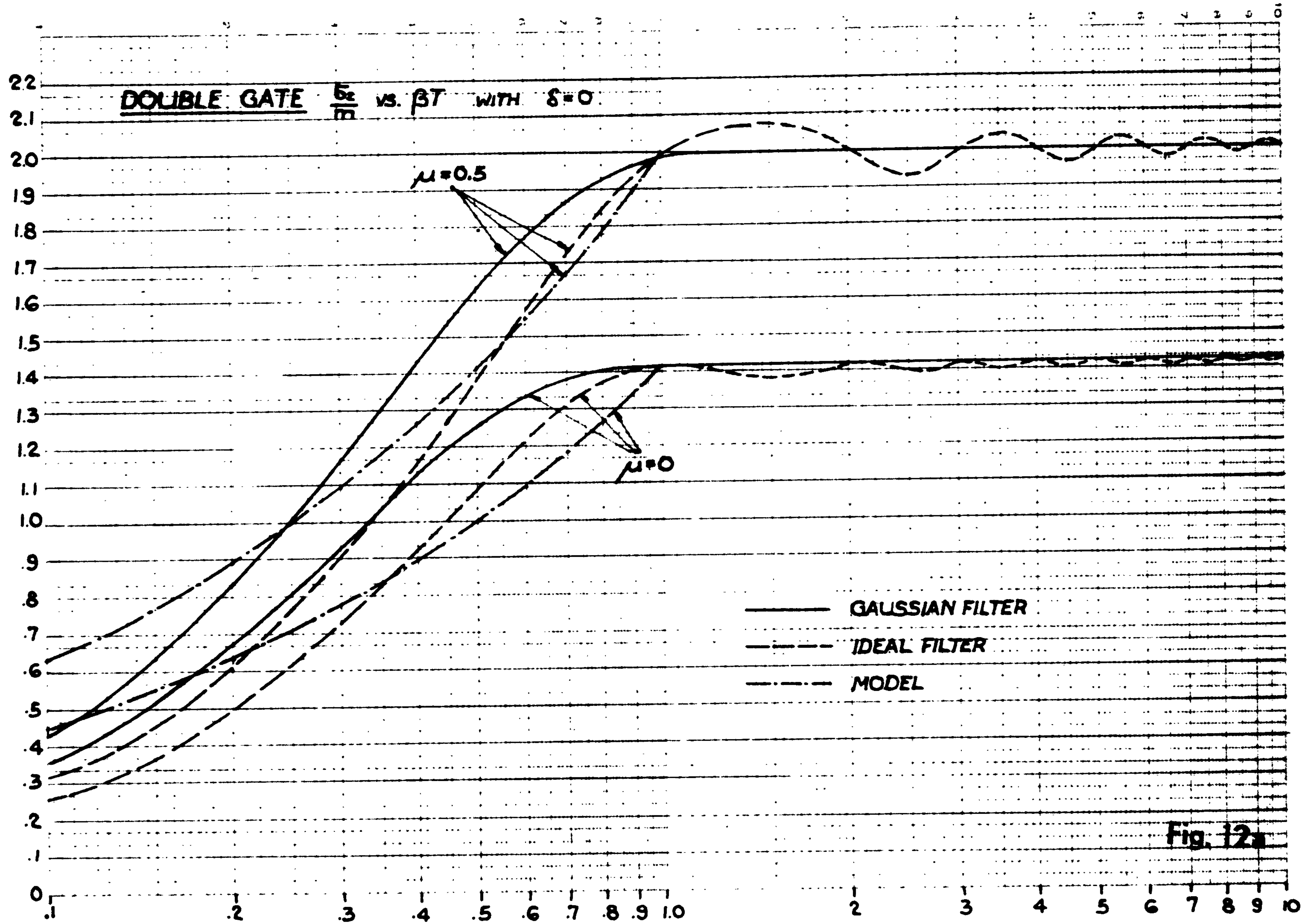

DOUBLE GATE $\frac{\overline{E_z}}{m}$ vs. βT WITH $\delta = 0$
$\mu = 0.5$
$\mu = 0$
GAUSSIAN FILTER
IDEAL FILTER
MODEL
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 2 3 4 5 6 7 8 9 10

Fig. 12a

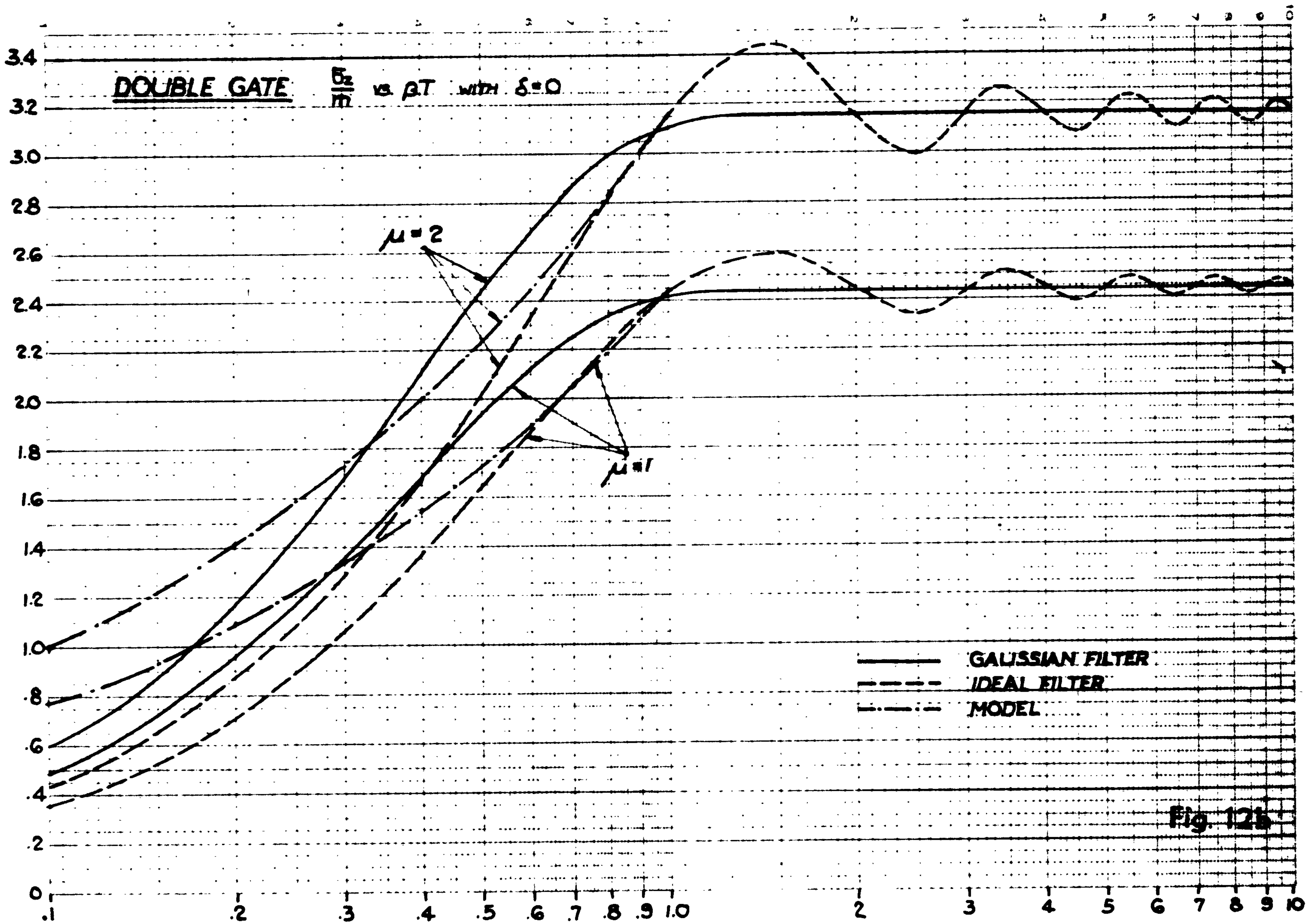

DOUBLE GATE $\frac{\bar{\sigma}_z}{m}$ vs. βT with $\delta = 0$
$\mu = 2$
$\mu = 1$
GAUSSIAN FILTER
IDEAL FILTER
MODEL
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
2
3
4
5
6
7
8
9
10
Fig. 12b

TABLE I

Values of $\frac{\sigma_y^2}{m^2}$ for $\mu = 0, 0.5, 1, 2$

	$\beta\delta = 0$	$\beta\delta = 1$	$\beta\delta = 2$	$\beta\delta = 3$	$\beta\delta = 4$	$\beta\delta = 5$
$\mu = 0$						
Simplified Model	1	1	0.5000	0.3333	0.2500	0.2000
Model	1	0.6667	0.4167	0.2963	0.2292	0.1867
Ideal	1	0.6559	0.3961	0.2825	0.2196	0.1797
Gaussian	1	0.5479	0.3225	0.2184	0.1665	0.1341
$\mu = 0.5$						
Simplified Model	2	2	1.0000	0.6667	0.5000	0.4000
Model	2	1.3333	0.8333	0.5926	0.4583	0.3733
Ideal	2	1.4296	0.8476	0.5929	0.4571	0.3716
Gaussian	2	1.2312	0.7431	0.5165	0.3968	0.3217
$\mu = 1.0$						
Simplified Model	3	3	1.5000	1.0000	0.7500	0.6000
Model	3	2.0000	1.2500	0.8889	0.6875	0.5600
Ideal	3	2.2033	1.2990	0.9032	0.6946	0.5634
Gaussian	3	1.9144	1.1637	0.8145	0.6272	0.5092
$\mu = 2.0$						
Simplified Model	5	5	2.5000	1.6667	1.2500	1.0000
Model	5	3.3333	2.0833	1.4815	1.1458	0.9333
Ideal	5	3.7507	2.2019	1.5239	1.1696	0.9471
Gaussian	5	3.2809	2.0049	1.4106	1.0879	0.8843

Appendix

In this appendix we give the details of the integrations involved in the determination of σ_y^2 and σ_z^2.

Let us consider first the ideal filter. Substituting (10) and (13) in (8) we have, for the square law detector, single gate,

$$\sigma_y^2 = 2\int_0^{\delta} (\delta - \tau)\left[4WP^2 \frac{\sin \pi \beta \tau}{\pi \beta \tau} + 4W^2 \left(\frac{\sin \pi\beta\tau}{\pi\beta\tau}\right)^2\right] d\tau \qquad \text{(A-1)}$$

To perform the integrations it was found convenient to introduce the function

$$G(x) = \int_0^x ds \int_0^s \frac{1-\cos t}{t^2} dt \qquad \text{(A-2)}$$

defined by

$$G''(x) = \frac{1-\cos x}{x^2} \quad , \qquad G'(o) = G(o) = 0 \qquad \text{(A-3)}$$

and in terms of which

$$\frac{\sin x}{x} = 2G''(x) + xG''(x)$$

$$\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} = \frac{2(1 - \cos x)}{x^2} = 2G''(x) \quad .$$

Thus, making the necessary substitution of variables in (A-1) we have

$$\sigma_y^2 = \frac{8WP^2}{(\pi\beta)^2}\int_0^{\pi\beta\delta} (\pi\beta\delta - x)\,[\,2G''(x) + xG''(x)\,]\,dx \qquad \text{(A-4)}$$

$$+ \frac{4W^2}{(\pi\beta)^2}\int_0^{2\pi\beta\delta} (2\pi\beta\delta - x)G''(x)ds$$

Integrating successively by parts and noting $G'(o) = G(o) = 0$ we then obtain

$$\sigma_y^2 = \frac{4W^2}{(\pi\beta)^2}\,[\,G(2\pi\beta\delta) + 4\,\mu\pi\beta\delta \quad G'(\pi\beta\delta)\,] \qquad \text{(A-5)}$$

where $\mu = \frac{P^2}{2W}$ is the pre-detection signal to noise power ratio. The result for the ideal filter with linear detector, single gate and $\mu >> 1$ is obtained similarly, substituting (12) and (13) in (8), giving

$$\sigma_y^2 = 2 \int_0^{\delta} (\delta - \tau) W \frac{\sin \pi\beta\tau}{\pi\beta\tau} \, d\tau \qquad \text{(A-6)}$$

$$= \frac{2W}{(\pi\beta)^2} \cdot \pi\beta\delta \; G'(\pi\beta\delta)$$

The results for the ideal filter with the double gate may be obtained in similar fashion. Substituting (10) and (13) in (9), we have, for the square law detector,

$$\sigma_z^2 = 4 \int_0^{\delta} [\delta - \tau] \left[4WP^2 \frac{\sin \pi\beta\tau}{\pi\beta\tau} + 4W^2 \left(\frac{\sin \pi\beta\tau}{\pi\beta\tau}\right)^2\right] d\tau$$

$$- \int_{T-\delta}^{T} [\tau - (T-\delta)] \left[4WP^2 \frac{\sin \pi\beta\tau}{\pi\beta\tau} + 4W^2 \left(\frac{\sin \pi\beta\tau}{\pi\beta\tau}\right)^2\right] d\tau \qquad \text{(A-7)}$$

$$-2 \int_{T}^{T+\delta} [T+\delta - \tau] \left[4WP^2 \frac{\sin \pi\beta\tau}{\pi\beta\tau} + 4W^2 \left(\frac{\sin \pi\beta\tau}{\pi\beta\tau}\right)^2\right] d\tau$$

$$= \frac{8W^2}{(\pi\beta)^2} \left[\begin{array}{l} G(2\pi\beta\tau) + 4\mu\pi\beta\delta\, G'(\pi\beta\delta) \\ + G(2\pi\beta T) + 4\mu\pi\beta T G'(\pi\beta T) \end{array}\right]$$

$$\frac{4W^2}{(\pi\beta)^2} \left[\begin{array}{l} G(2\pi\beta(T-\delta)) + 4\mu\pi\beta(T-\delta) G'(\pi\beta(T-\delta)) \\ + G(2\pi\beta(T+\delta) + 4\mu\pi\beta(T+\delta) G'(\pi\beta(T+\delta)) \end{array}\right]$$

For the linear detector, $\mu >> 1$, we have, substituting (12) and (13) in (9)

$$\sigma_z^2 = 4\int_0^{\delta}[\delta-\tau]\; W\,\frac{\sin\pi\beta\tau}{\pi\beta\tau}\; d\tau$$

$$-\,2\int_{T-\delta}^{T}[\tau-(T-\delta)]\; W\,\frac{\sin\pi\beta\tau}{\pi\beta\tau}\; d\tau \tag{A-8}$$

$$-\,2\int_{T}^{T+\delta}[T+\delta-\tau]\; W\,\frac{\sin\pi\beta\tau}{\pi\beta\tau}\; d\tau$$

$$= \frac{4W}{(\pi\beta\;)^2}\;[\pi\beta\delta G'(\;\pi\beta\delta\;) + \pi\beta\;\;T\,G'(\;\pi\beta\;T)]$$

$$-\,\frac{2W}{(\pi\beta\;)^2}\;[\pi\beta\;(T-\delta)G'(\pi\beta\;(T-\delta)) + \pi\beta\;\;(T+\delta)G'(\pi\beta\;\;(T+\delta)]$$

We consider next the Gaussian filter. Substituting (10) and (14) in (8) we have, for the square law detector, single gate

$$\sigma_y^2 = 2\int_0^{\delta}(\delta-\tau)\;[4WP^2\,e^{-\pi\beta^2\tau^2} + 4W^2\,e^{-2\pi\beta^2\tau^2}]\;\; d\tau \tag{A-9}$$

$$= \frac{8WP^2}{\pi\beta^2}\int_0^{\beta\delta\sqrt{\pi}}(\,\beta\delta\,\sqrt{\pi}\;\;-x)e^{-x^2}dx + \frac{4W^2}{\pi\beta^2}\int_0^{\beta\delta\sqrt{2\pi}}(\beta\delta\;\;\sqrt{2\pi}\;-x)e^{-x^2}\,dx$$

where we have made the substitution $x = \beta\tau\sqrt{\pi}$ to obtain the first integral and $x = \beta\tau\sqrt{2\pi}$ to obtain the second. The integrations are now straightforward and may be expressed in terms of the function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-t^2}\,dt \tag{A-10}$$

The results for the Gaussian filter with the double gate may be obtained in similar fashion. The particular form chosen in writing the terms in the expressions for σ_y^2 and σ_z^2 given in the text was

the one found most convenient for the purpose of computation.

In the case of the model, (15) (for the square law detector) and (16) (for the linear detector, $\mu \gg 1$) must be substituted in (8) and (9) to obtain σ_y^2 and σ_z^2, respectively. The expressions in the integrands are second degree polynomials and hence the integrations present no difficulties. However, several cases must be distinguished with the model, depending on the values of $\beta\delta$ and βT, since the autocorrelation function minus its d.c. value is zero for $\beta |\tau| \geqq 1$. For the single gate we must consider $\beta\delta \leqq 1$ and $\beta\delta > 1$. In performing the integrations for the double gate we must consider the following six cases:

1) $\beta\delta < 1$
   $\beta(T + \delta) < 1$

2) $\beta\delta < 1$
   $\beta T < 1 \leqq \beta(T + \delta)$

3) $\beta\delta < 1$
   $\beta(T - \delta) < 1 \leqq \beta T$

4) $\beta\delta < 1$
   $\beta(T - \delta) \geqq 1$

5) $\beta\delta \geqq 1$
   $\beta(T - \delta) < 1 \leqq \beta T$

6) $\beta\delta \geqq 1$
   $\beta(T - \delta) \geqq 1$

In order to obtain σ_y^2 and σ_z^2 in the limit of $\beta\delta$ or βT very large or very small, it is useful to note that for small x,

$$G(x) \sim \frac{x^2}{4}$$

$$G'(x) \sim \frac{x}{2} \qquad \text{from (A-3)} \qquad \text{(A-11)}$$

and

$$\mathrm{Erf}(x) \sim \frac{2x}{\sqrt{\pi}} \qquad \text{from (A-10)}$$

For large x we have (cf. Ref. A, p. 222, Eq. (8.8))

$$G'(x) \sim \frac{\pi}{2} x - (\log x + 1 + C)$$

$$G'(x) \sim \frac{\pi}{2} \qquad \text{(A-12)}$$

where C = Euler's constant = 0.5772 . . .

and $\mathrm{Erf}(x) \sim 1$.

As has been noted earlier, comparison may be made with the results of Rice (Ref. A) for noise alone in the case of the ideal filter, single and double gate and in the case of the Gaussian filter, single gate. For the ideal filter, single gate, the result is given in Ref. A, p. 223, Eq. (8.10). Noting that where Rice uses σ_T , βw_0 , $f_b - f_a$ and $x^2F(x)$ we use σ_y , W, δ, β and G(x), respectively, we may set $\mu = 0$ in (17), which is then exactly four times the value given by Rice, as it should be. (See comment in paragraph beginning on line 5, page 13.) For the ideal filter with double gate the results in Ref. A are given on p. 224, Eq. (7.3), in the form of an infinite series which may be expressed in terms of F(x), as noted there. Using Eq. (6.7) of Ref. A and noting that $\sigma_{S,T}$, S,T of Ref. A correspond to σ_z , T , δ , respectively of this paper, one again finds that (after setting $\mu = 0$ in (28)), σ_z^2 is four times $\sigma^2_{S,T}$ as given in Ref. A.

Finally, the result for the Gaussian filter, single gate, noise

alone is given in Ref. A (p. 225) in the form of an infinite series. This series may be summed as shown below and the result compared with (18) for the particular case $\mu = 0$.

From p. 225 of Ref. A we have

$$\sigma_T^2 = w_o^2 \frac{a^2}{2\pi} \sum_{n=o}^{\infty} \frac{(-1)^n a^{2n}}{(n+1)!(2n+1)}$$

where $a = \beta\delta \sqrt{2\pi}$

$$w_o = \frac{W}{\beta}$$

in the notation we have been using. Thus, if we define

$$f(a) = \frac{\sigma_T^2}{W^2 \delta^2} = \sum_{n=o}^{\infty} \frac{(-1)^2 a^{2n}}{(n+1)!(2n+1)}$$

then

$$[af(a)]' = \sum_{n=o}^{\infty} \frac{(-1)^n a^{2n}}{(n+1)!}$$

or

$$a^2[af(a)]' = -\sum_{n=o}^{\infty} \frac{(-a^2)^{n+1}}{(n+1)!} = 1 - \sum_{n=o}^{\infty} \frac{(-a^2)^n}{n!}$$

$$= 1 - e^{-a^2}$$

Integrating, we have

$$af(a) = \int_o^a \frac{1-e^{-t^2}}{t^2}\, dt = \int_o^a (1-e^{-t^2})\, d(-\frac{1}{t})$$

$$= -\left(\frac{1-e^{-t^2}}{t}\right)\Bigg]_o^a + 2\int_o^a e^{-t^2}\, dt$$

$$= \sqrt{\pi}\ \mathrm{Erf}(a) + \frac{e^{-a^2}-1}{a}$$

or

$$f(a) = \frac{\sigma_T^2}{W^2 \delta^2} = \frac{\sqrt{\pi}}{a} \mathrm{Erf}(a) + \frac{e^{-a^2}-1}{a^2}$$

Substituting $a = \beta\delta \sqrt{2\pi}$, we have

$$\sigma_T^2 = \frac{W^2}{4\beta^2\sqrt{\pi}} \left[2\,\beta\delta\,\sqrt{2\,\pi}\;\mathrm{Erf}(\beta\,\delta\,\sqrt{2\,\pi}) + \right.$$

$$\left. + \frac{2}{\sqrt{\pi}}\; e^{-2\,\pi\beta^2\,\delta^2} - \frac{2}{\sqrt{\pi}} \right]$$

which is one fourth of σ_y^2 as given in (18) with $\mu = 0$, as it should be.

TABLE II

Values of G(x) and G'(x)*

x	G(x)	x	G'(x)	x	G'(x)
0	0	0	0	14.5	1.4973
.2	+ .0100	.1	.04996	15.0	1.5009
.4	.0399	.2	.09989	15.5	1.5049
.6	.0896	.4	.1991	16.0	1.5090
.8	.1586	.6	.2970	16.5	1.5122
1.0	.2466	.8	.3930	17.0	1.5151
1.5	.5455	1.0	.4864	17.5	1.5169
2.0	.9473	1.2	.5767	18.0	1.5177
2.5	1.4375	1.5	.7052	19.0	1.5180
3.0	1.9998	1.7	.7856	19.5	1.5182
3.5	2.6174	2.0	.8973	20.0	1.5186
4.0	3.2747	2.2	.9656	21.0	1.52[illegible]
4.5	3.9581	2.5	1.0581	22.0	1.5[illegible]
5.0	4.6566	2.8	1.1384	23.0	1.52[illegible]
6.0	6.0713	3.0	1.1853	24.0	1.5308
7.0	7.4897	3.5	1.2798	25.0	1.5311
8.0	8.9138	3.7	1.3091	30.0	1.5352
9.0	10.3552	4.0	1.3448	48.0	1.55[illegible]8
10.0	11.8192	4.2	1.3636	50.0	1.55[illegible]
11.0	13.301	4.5	1.3851	60.0	1.55500
12.0	14.7916	4.8	1.3999		
13.0	16.284	5.0	1.4067		
14.0	17.777	5.5	1.4158		
15.0	19.2743	6.0	1.4180		
16.0	20.779	6.5	1.4182		
17.0	22.292	7.0	1.4194		
18.0	23.808	7.5	1.4236		
19.0	25.326	8.0	1.4310		
19.5	26.085	8.5	1.4411		
20.0	26.844	9.0	1.4527		
24.0	33.425	9.5	1.4642		
28.0	39.071	10.0	1.4744		
30.0	42.168	10.5	1.4824		
36.0	51.387	11.0	1.4878		
48.0	69.952	11.5	1.4908		
50.0	73.0619	12.0	1.4920		
60.0	88.576	12.5	1.4922		
100.0	150.8978	13.0	1.4922		
		13.5	1.4929		
		14.0	1.4946		

*Note Eqs. (A-2), (A-3), (A-11), (A-12).